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A Mathematica module for two-dimensional computer graphics -Data structure and Interpolation algorithms-

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Abstract

2D shape interpolation is widely used in Computer Graphics. We introduce a Mathematica module for drawing, transformation, interpolation of two-dimensional polygon figure using results [1] and [2]. We can analyse and investigate critical examples of interpolations using our module. Symbolic computations in Mathematica enable us to evaluate those examples using several mathematical formulas. We can use our functions to produce a pre-expanded formulas to compute an interpolation in another language such as C which does not have a facility of symbolic computations.

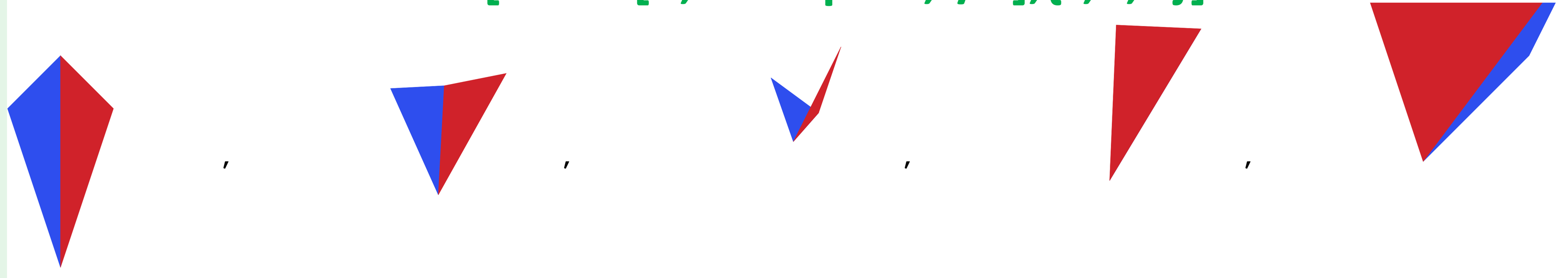
2D Shape Interpolation

Interpolating a source and target figures of 2D polygons.

Examples

Alexa + Alexa Error + C2

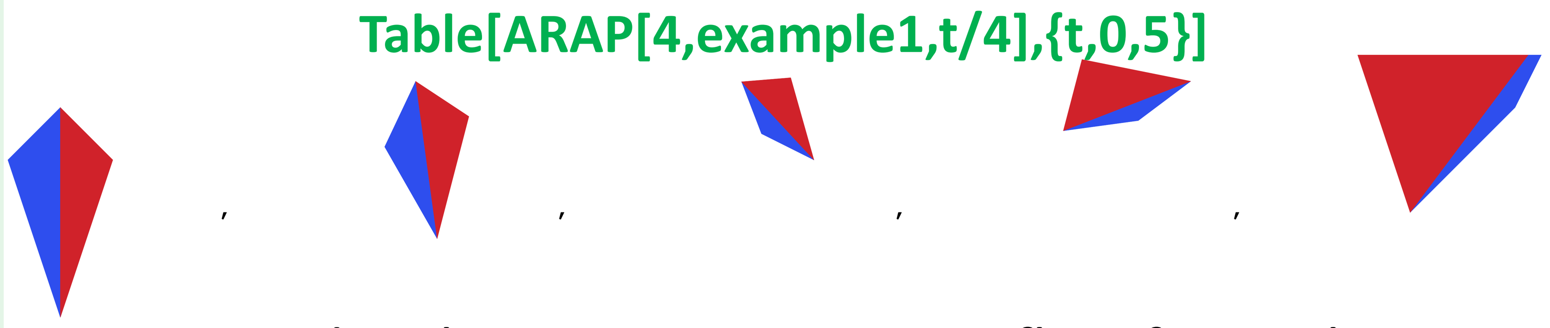
Table[ARAP[2,example1,t/4],{t,0,5}]



The triangle collapses and then turns over.

Alexa + Sim Error+ C2

Table[ARAP[4,example1,t/4],{t,0,5}]



Rotation and scale invariance prevents flip of triangles.

Local Interpolations

Interpolation of each triangle.[1][2]

- Linear Interpolation $A^L(t) := (1-t)E + tA$
  Each point move in a straight line.
- Alexa's Interpolation $A^P(t) = R_{t\theta}((1-t)E) + tS)$
  Rotation Matrix × Symmetric Matrix(Linear)
- Log-Exp Interpolation $A^E(t) = R_{t\theta}\exp(t\log S)$
  Rotate Matrix × Symmetric Matrix

Error functions

It measures how different local and global interpolation are as linear maps(Alexa)/linear maps up to rotation and scale(Sim).[1][3][4][5][6]

- Alexa's Error $E_k^F(B_k, A_k(t)) := ||B_k - A_k(t)||_F^2$
- Sim Error $E_k^R(B_k, A_k(t)) := ||B_k||_F^2 - \frac{||B_k \cdot A_k^T||_F^2 + 2\det(B_k \cdot A_k^T)}{||A_k||_F^2}$

Constraint functions

It controls the global interpolation.[2]

- $C1(v(t)) = ||(1-t)p_1 + tq_1 - v_1(t)||^2$
- $C2(v(t)) = ||(1-t)p_1 + tq_1 - v_1(t)||^2 + ||(1-t)p_2 + tq_2 - v_2(t)||^2$
- $C3(v(t)) = ||(1-t)p_m + tq_m - v_m(t)||^2$
- $C4(v(t)) = ||v_k(t) - v_l(t) - e_{kl}(t)||$ $\quad (e_{kl}(t) = e_l(t) - e_l(t))$
- $C5(v(t)) = ||v_k(t) - v_l(t) - e_{kl}'(t)||$ $\quad (e_{kl}'(t) = R(2\pi t)e_{kl}(t))$

Global Error Functions

- $E_F(t) = \sum_{k\in\Delta} ||B_k - A_k(t)||_F^2 + C(v(t))$
- $E_S(t) = \sum_{k\in\Delta} ||B_k||_F^2 - \frac{||B_k \cdot A_k^T||_F^2 + 2\det(B_k \cdot A_k^T)}{||A_k||_F^2} + C(v(t))$

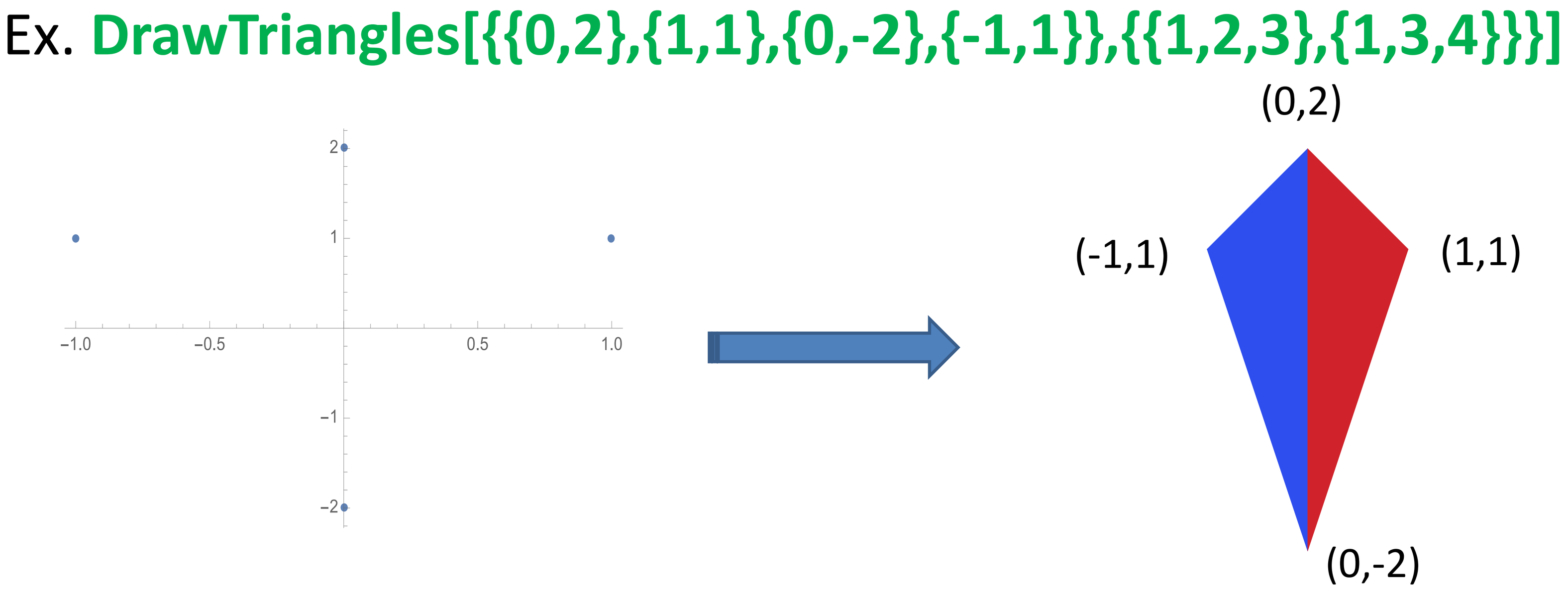
Goal

Compute v(t) that minimizes the Global Error Function.

The module can …

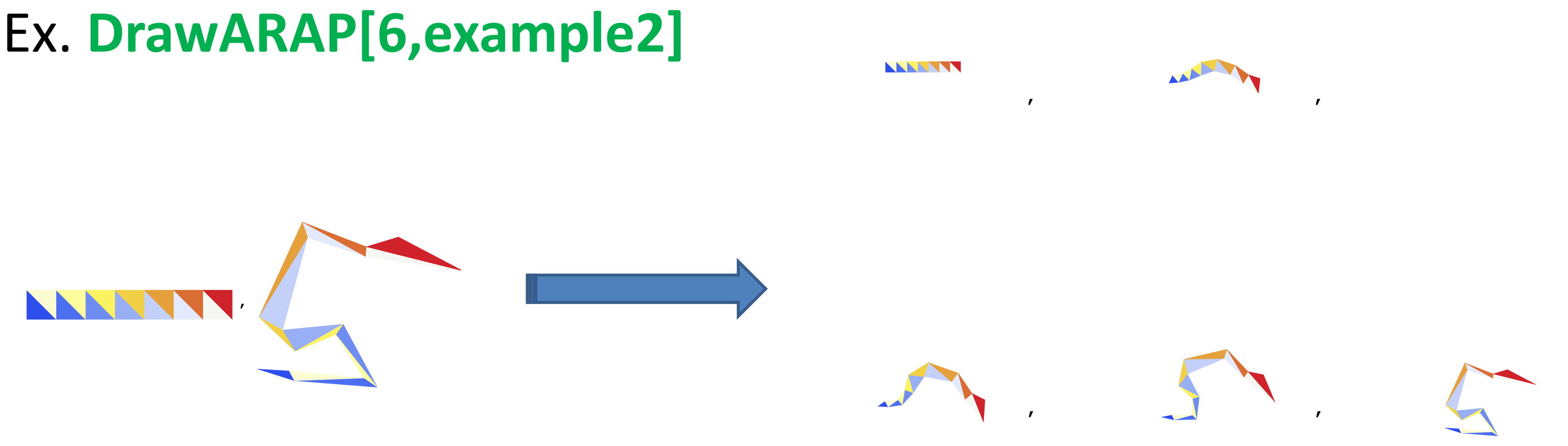
- Draw triangled polygons assigning colors.

Ex. DrawTriangles[{{0,2},{1,1},{0,-2},{-1,1}},{{1,2,3},{1,3,4}}]



- Make an interpolation from a source and a target figures.

Ex. DrawARAP[6,example2]



Speeding up using pre-expanded formulas

- We note that every error function is a positive quadratic form in elements of v(t).In order to have a unique minimizer v(t), we need some constraints. The minimization problem is solved as an inverse matrix of a quadratic coefficients matrix of an global error function. Our module can make a symbolic representation of the quadratic coefficients matrix.
- The case of using Alexa's error, the matrix is defined time-independent, so we only need to compute them just once. The case of using Sim error, the matrix is depend on time, but our module give a symbolical expansions of a function which can be used in another efficient language as a hard coding.

Conclusion

- We developed a Mathematica module for checking an effect of 2D-ARAP interpolations easily.
- We can use our functions to produce a pre-expanded formulas to compute an interpolation in C programming language.
- Future works include to extend our module for 3D CG and develop interface to other CG software such as Maya.
- This module have been published on GitHub with its manual. https://github.com/KyushuUniversityMathematics/MathematicaARAP

References

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