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Origami System using Conformal Geometric Algebra

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Abstract

In 2006 , Ida formalized a 2D origami for developing the system *Eos*[1] for proving theorems about origami properties. In 2014 , Ida introduced an idea to extend *Eos* for 3D origami. We developed CGA library using *Matematica*[2]. Our motivation is to realize 3D origami system using our *Mathematica* CGA Library developed in 2014. To this end , we implemented a *Mathematica* module of the 2D origami system which includes the origami data structure and the fold operation function. In addition, we formulated origami by using GA points and simple fold operations by using GA motions. We can prove geometric theorems of origami by calculating GA equations. Our future work includes finding a 3D origami motion which is useful for a computer graphics animation.

Introduction

We are making a computational 3D origami system by formulating origami and fold operation in GA.

In 2D, a fold operation can be represented by a fold line m and a mountain or valley fold.

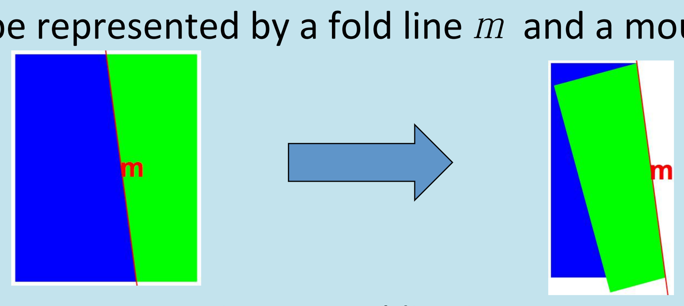
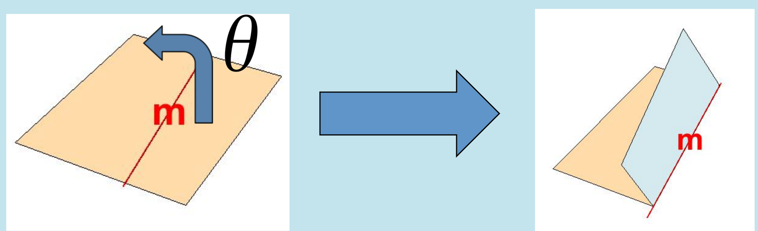


Figure 1: Folding in 2D

In 3D, a fold operation can be represented by a fold line m and an angle θ.



We can express a fold operation in a GA element.

Figure 2: Folding in 3D

advantage : We have a simple expression of fold operation.
We can prove theorems in origami.

5-dimensional conformal model

- We use *5-dimensional conformal model.*
- This model has five bases.
  - — Standard basis {$\mathbf{w}_{\{1\}}$, $\mathbf{w}_{\{2\}}$, $\mathbf{w}_{\{3\}}$} for the 3D space $\mathbf{R}^3$.
  - — Null basis $\mathbf{w}_{\{0\}}$ corresponds to the origin point.
  - — Null basis $\mathbf{w}_{\{\infty\}}$ corresponds to the infinity point.
- This model is a 32-dimensional linear space ($2^5 = 32$).
- We consider basis $\mathbf{w}_S$ for subset S of {0, 1, 2, 3, ∞}.
- Our module can denote CGA elements.
- Our module can compute several operations in CGA.
  - — Operator ^ is the **outer product**.
  - — Operator ・ is the **inner product**.

・	$w_{\{1\}}$	$w_{\{2\}}$	$w_{\{3\}}$	$w_{\{0\}}$	$w_{\{\infty\}}$
$w_{\{1\}}$	1	0	0	0	0
$w_{\{2\}}$	0	1	0	0	0
$w_{\{3\}}$	0	0	1	0	0
$w_{\{0\}}$	0	0	0	0	-1
$w_{\{\infty\}}$	0	0	0	-1	0

Table 1: Our CGA basis.

・Point $P_a(x_a, y_a, z_a) = w_{\{0\}} + x_a w_{\{1\}} + y_a w_{\{2\}} + z_a w_{\{3\}} + \frac{x_a^2 + y_a^2 + z_a^2}{2} w_{\{\infty\}}$

CGA Motions

CGA have some motions. The next figure is an example figure of the CGA operation and the algebraic formula which corresponds to that.

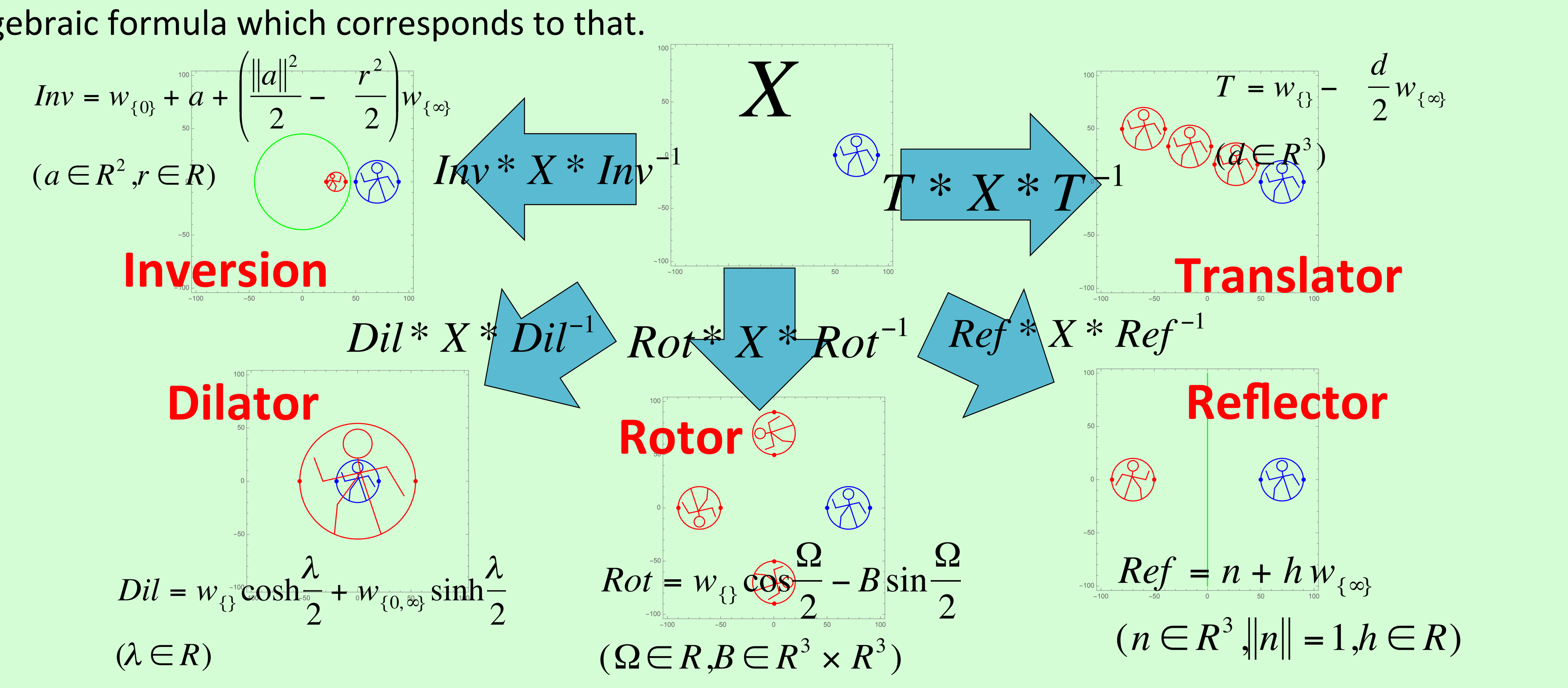


Figure 3:CGA motions.

Origami data structure

Origami graph $O = (\Pi, \sim, \succ)$ has three elements **Face set** Π , **Adjacency relation** $\sim$, **Superposition relation** $\succ$.

- $\Pi = \{ f_1, f_2, \cdots, f_i, \cdots, f_n \}$ is the **Face set**.
  - — Where $f_i = \{ p_1^i, p_2^i, \cdots, p_m^i \} \in \Pi$ is vertex coordinates sets.
  - — For example , $\Pi = \{ f_4, f_5, f_6, f_7 \}$ (Figure 4).

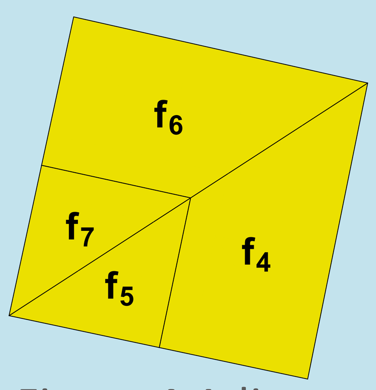


Figure 4:Adjacency

- $\sim$ is the **Adjacency relation**.
  - — For example , $\sim = \{ (f_4, f_5), (f_4, f_6), (f_5, f_7), (f_6, f_7) \}$ (Figure 4).

- $\succ$ is the **Superposition relation**.
  - — $(f, g) \in \succ$ superposed "directly".
  - — If we know superposition relations between two faces by combined other ones , we simplify this relation by deleting its column.
  - — For example , $\succ = \{ (f_7, f_6), (f_6, f_4), (f_4, f_5) \}$ (Figure 5) , however $(f_7, f_5) \notin \succ$.

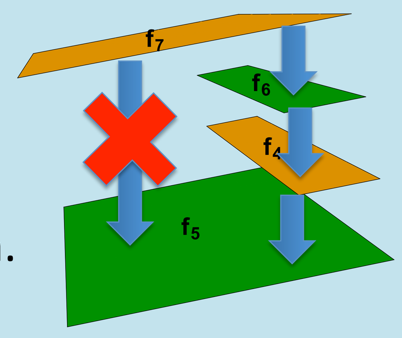


Figure 5:Superposition

Acknowledgments

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2D Origami folding

Function *Ori* is the fold operation algorithm in our module.

Input: Origami graph $O = (\Pi, \sim, \succ)$, fold line m , folded faces sets F , angle $\theta \in \{ \pi , -\pi \}$

— We can calculate fold line m by *Huzita–Hatori axioms*.

"Divide operation" $\{O = (\Pi, \sim, \succ), m , F \} \Rightarrow \{ O' = (\Pi', \sim', \succ'), D \}$

$C \supset F$: the face set which is folded just as F is folded due to adjacency relation

$\Pi' : f_i \in \Pi \to f_i \in \Pi' \quad (f_i \in C \text{ isn't divided})$

$f_{2i}, f_{2i+1} \in \Pi' \quad (f_i \text{ is divided})$ Then f_{2i+1} is in the folded part.

$\sim' : (f_i, f_j) \in \sim \to (f_i, f_j) \in \sim' \quad (f_i, f_j \text{ aren't divided})$

$(f_i, f_{2j+1}) \in \sim' \quad (f_i \in C \text{ isn't divided}, f_j \text{ is divided})$

$(f_i, f_{2j}) \in \sim' \quad (f_i \notin C \text{ isn't divided}, f_j \text{ is divided})$

$(f_{2i}, f_{2j}), (f_{2i+1}, f_{2j+1}) \in \sim' \quad (f_i, f_j \text{ are divided})$

$f_i \in \Pi \to (f_{2i}, f_{2i+1}) \in \sim' \quad (f_i \text{ is devided})$

$\succ' : (f_i, f_j) \in \succ \to (f_i, f_j) \in \succ' \quad (f_i, f_j \text{ aren't divided})$

$(f_i, f_{2j}), (f_i, f_{2j+1}) \in \succ' \quad (f_i \text{ isn't divided}, f_j \text{ is divided})$

$(f_{2i}, f_j), (f_{2i+1}, f_j) \in \succ' \quad (f_i \text{ is divided}, f_j \text{ isn't divided})$

$(f_{2i}, f_{2j}), (f_{2i+1}, f_{2j+1}) \in \succ' \quad (f_i, f_j \text{ are divided})$

But, if $f_i, f_j \in \Pi'$ don't overlap, $(f_i, f_j) \notin \succ'$.

$D : C \to D = \{ f \in \Pi' \mid f \in C \text{ isn't divided} \} \cup \{ f_{2i+1} \in \Pi' \mid f_i \in C \text{ is divided} \}$

"Rotate operation" $\{O = (\Pi, \sim, \succ) , m , \theta , D \} \Rightarrow \{ O'' = (\Pi'', \sim', \succ'') \}$

R : the GA operation corresponds to the rotate operation

$\Pi_M \supset D$: the face set which is folded just as D is folded due to adjacency and superpose relations

We judge the possibility of folding by checking the relation m and Π_M.

$\Pi'' : f_i \in \Pi' \to f_i \in \Pi'' \quad (f_i \notin \Pi_M), R * f_i * R^{-1} \in \Pi'' \quad (f_i \in \Pi_M)$

$\succ'' : (f_i, f_j) \in \succ' \to (f_i, f_j) \in \succ'' \quad (f_i, f_j \notin \Pi_M), (f_j, f_i) \in \succ'' \quad (f_i, f_j \in \Pi_M)$

If $\theta = \pi, f_i \in \Pi_M, f_j \notin \Pi_M \to (f_i, f_j) \in \succ'' \quad (f_i, f_j \in \Pi'' \text{ overlap and superposed directly})$

If $\theta = -\pi, f_i \notin \Pi_M, f_j \in \Pi_M \to (f_i, f_j) \in \succ'' \quad (f_i, f_j \in \Pi'' \text{ overlap and superposed directly})$

Output: Origami graph $O'' = (\Pi'', \sim', \succ'')$

Proof and GA equations

We make various figures by folding an origami. We can prove its geometric proposition by calculating GA equations.

For example, when we fold a quadratic prism from rectangular origami as follows (Figure 6), we show point 1 and 2 are in the same position.

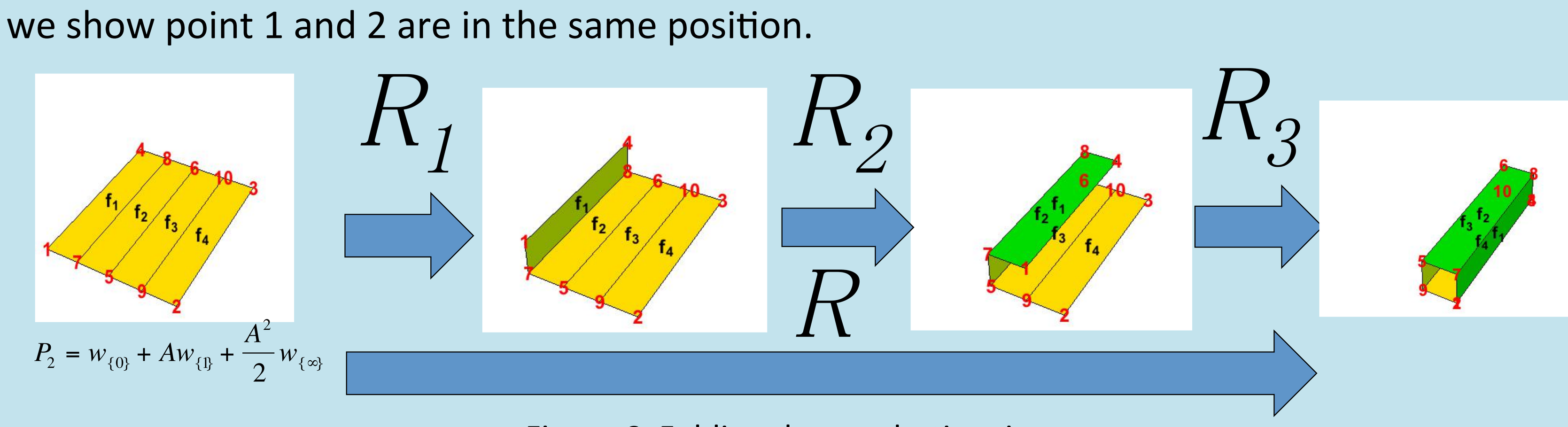

Figure 6: Folding the quadratic prism

We can express fold operations R_1, R_2, R_3 in GA and the GA operation R combined R_1, R_2, R_3.

$$R_1 = \frac{w_{\{\}}}{\sqrt{2}} + \frac{\sqrt{B^2}}{\sqrt{2}B} w_{\{1,3\}} - \frac{A\sqrt{B^2}}{4\sqrt{2}B} w_{\{3,\infty\}},\ R_2 = \frac{w_{\{\}}}{\sqrt{2}} + \frac{\sqrt{B^2}}{\sqrt{2}B} w_{\{1,3\}} - \frac{A\sqrt{B^2}}{2\sqrt{2}B} w_{\{3,\infty\}},\ R_3 = \frac{w_{\{\}}}{\sqrt{2}} + \frac{\sqrt{B^2}}{\sqrt{2}B} w_{\{1,3\}} - \frac{3A\sqrt{B^2}}{4\sqrt{2}B} w_{\{3,\infty\}},$$

$$R = R_3 * R_2 * R_1$$

$$= \left(\frac{w_{\{\}}}{\sqrt{2}} + \frac{\sqrt{B^2}}{\sqrt{2}B} w_{\{1,3\}} - \frac{3A\sqrt{B^2}}{4\sqrt{2}B} w_{\{3,\infty\}} \right) \left(\frac{w_{\{\}}}{\sqrt{2}} + \frac{\sqrt{B^2}}{\sqrt{2}B} w_{\{1,3\}} - \frac{A\sqrt{B^2}}{2\sqrt{2}B} w_{\{3,\infty\}} \right) \left(\frac{w_{\{\}}}{\sqrt{2}} + \frac{\sqrt{B^2}}{\sqrt{2}B} w_{\{1,3\}} - \frac{A\sqrt{B^2}}{4\sqrt{2}B} w_{\{3,\infty\}} \right)$$

$$= -\frac{w_{\{\}}}{\sqrt{2}} + \frac{\sqrt{B^2}}{\sqrt{2}B} w_{\{1,3\}} + \frac{A}{2\sqrt{2}} w_{\{1,\infty\}} - \frac{A\sqrt{B^2}}{2\sqrt{2}B} w_{\{3,\infty\}}$$

We can calculate the CG equation

$$R * P_1 * R^{-1} = w_{\{0\}} + A w_{\{1\}} + \frac{A^2}{2} w_{\{\infty\}} = P_2.$$

This equation correspond that point 1 and 2 are in the same position.

Future works

1. To define a superposition relation in 3D origami.
2. To implement the function to judge the collision of faces in 3D origami.
3. To implement traditional folds, i.e. inside reverse fold, outside reverse fold, etc.

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