

A certified Wang tiling program with the Coq proof assistant



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Abstract

We have implemented a certified Wang tiling program for tiling a rectangle region using a brick corner Wang tile set. A brick corner Wang tile set is a special Wang tile set being introduced by A. Derouet-Jourdan et al. for creating wall patterns texture[4]. We have implemented an algorithm using Coq proof assistant and proved its correctness. This correctness assures the existence of a tiling of any brick corner Wang tile set for any size of rectangle. The essential points of our proof is based on the existence of a tiling for a 2 times 2 rectangle and inductions. Since the brick corner Wang tile is a class of infinite kinds of tile sets, it is not straightforward and there are many conditional branches to prove the correctness. The certification with Coq assures that there are no lack of conditions.

Wang tiling

A Wang tile is a square tile with a color on each side. We arrange Wang tiles side by side with matching colors of edges. The followings are examples of a correct arranging and an incorrect one.

Correct: Incorrect:

We show an example of a tiling of the plane on (Figure 2). We note boundaries of tiles are vertical lines and horizontal lines.

There is a basic problem about a given set of Wang tiles whether it can tile the plane or not, which is introduced by Hao Wang in 1961.

Now we consider problems for a given tile set. Example in (Figure 2) is also using a tile set with 13 kinds of tiles shown on (Figure 1).

Wang tiling can be applied to computer graphics. (Figure 3) is a computer graphics of a sunflower field using Wang tiling (Cohen 2003)[2].

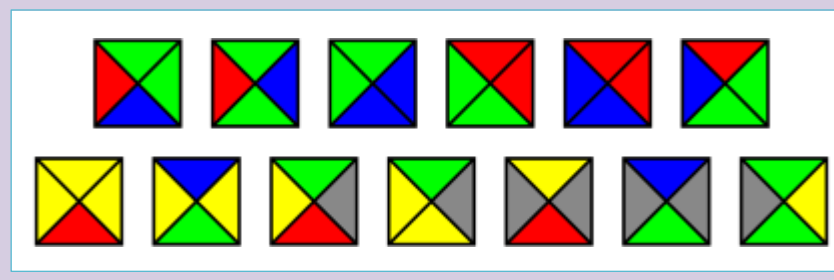


Figure 1 : Wang tiles.

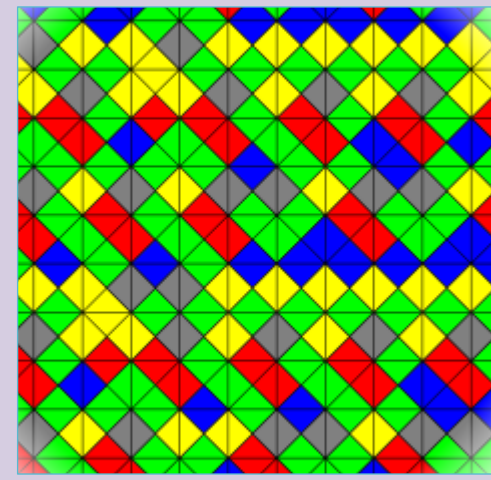


Figure 2 : Tiling.

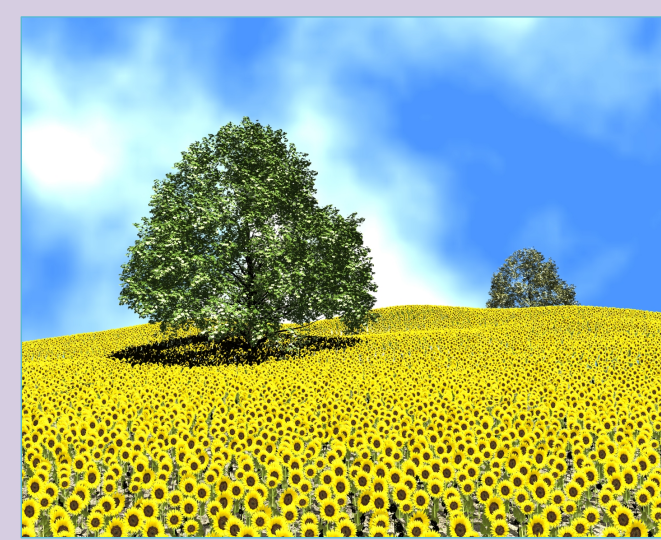


Figure 3 : Computer graphics.

Data structures

A valid tiling is defined by an edge coloring. Each Wang tile in a rectangle is specified a given edge coloring (Figure 7).

Then we define edge function e and e' like (Figure 8).

There are horizontal edges from $e(0, 1)$ to $e(n, m)$ and vertical edges from $e'(1, 0)$ to $e'(n, m)$.

A boundary condition is a function

$b : nat \rightarrow nat \rightarrow nat$.

For a valid Wang tiling, the followings holds

$e(0, j) = b(0, j)$, $e(n, j) = b(n+1, j)$, $e'(i, 0) = b(i, 0)$ and $e'(i, m) = b(i, m+1)$.

We note there is a technical adjustment of indices.

For example, in (Figure 8), $b(2, 3)$ can be assigned for both $e(2, 3)$ and $e'(2, 3)$.

We do use $b(2, 3+1)$ for right boundary $e'(2, 3)$ and $b(2+1, 3)$ for bottom boundary $e(2, 3)$.

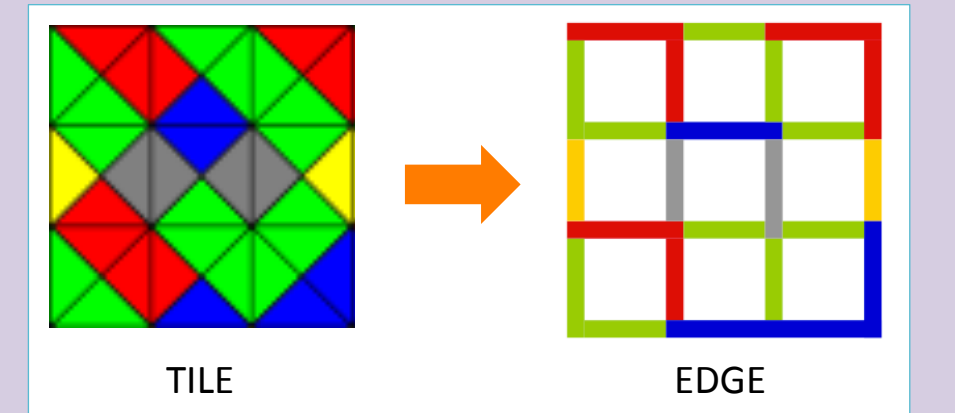


Figure 7 : We can consider Wang tiling as edge coloring.

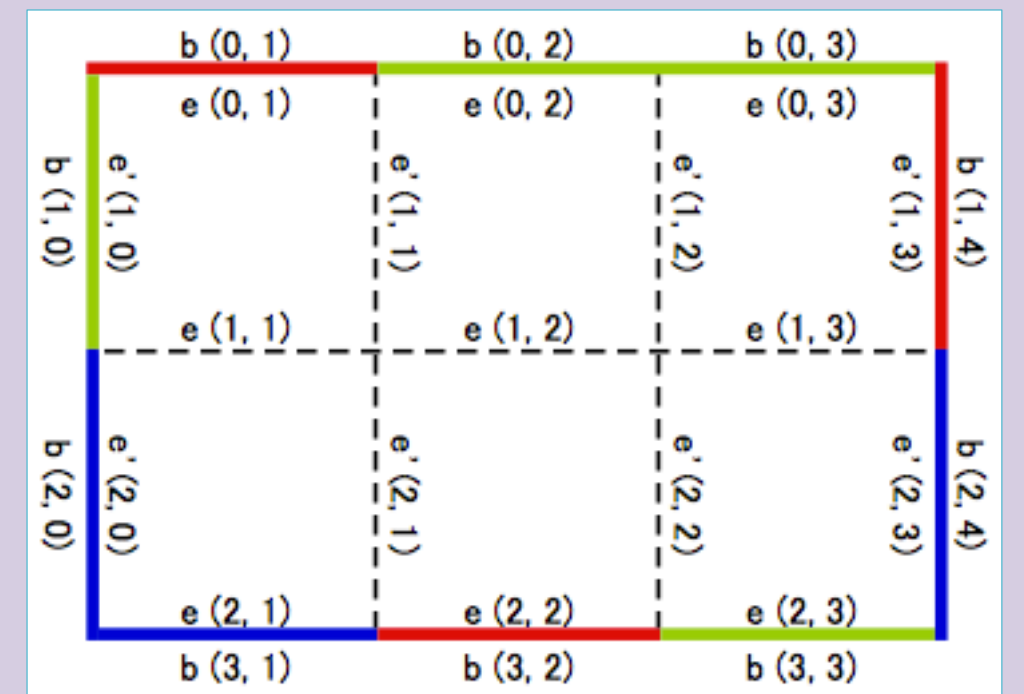


Figure 8 : Edge functions and boundaries.

Tiling algorithm and its result

Outline of our algorithm (Figure 9):

1. Choosing appropriate tiles and reducing the size of the tiling region to $2 \times m$.
2. Choosing appropriate tiles and reducing the size of the tiling region to 2×2 .
3. 2×2 problem can be proved by many kinds of conditional branches.

To execute a function in Coq we use *Compute* command. Our tiling function name is *tiling_nm*. Therefore, we input *Compute (tiling_nm n m b)*, where n and m are size of rectangles and $b : nat \rightarrow nat \rightarrow nat$ is a boundary condition. (Figure 10) is an example of the execution result.

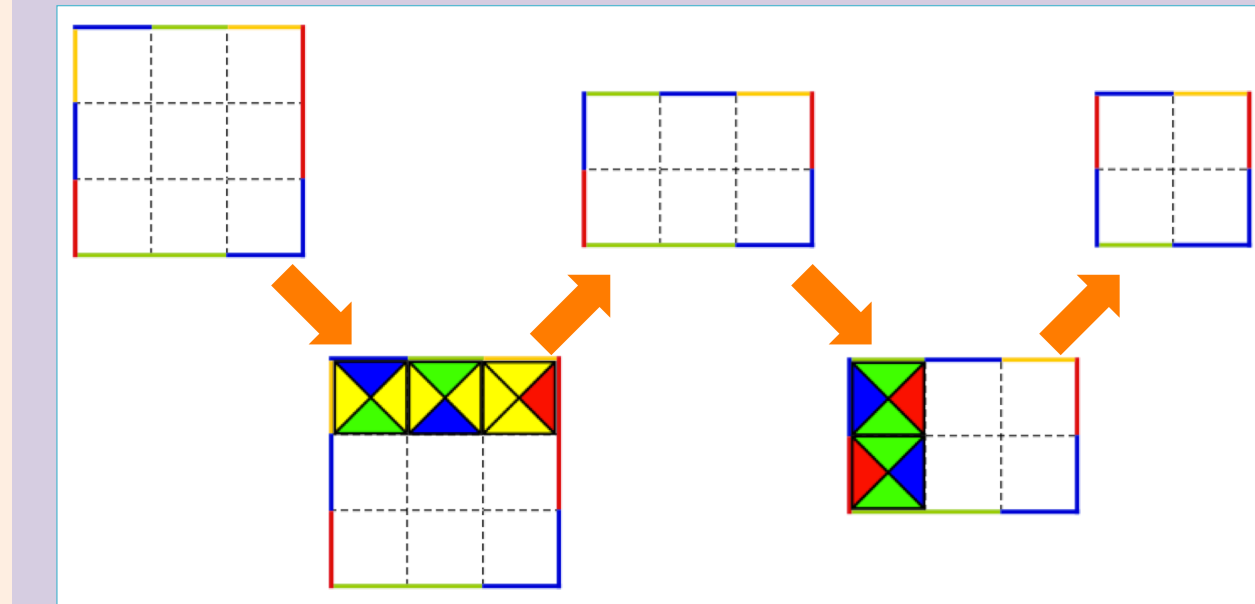


Figure 9 : Algorithm.

```
= (^ :: 2 :: ^ :: 0 :: ^ :: 1 :: ^ :: 0 :: ^ :: nil)
:: (2 :: # :: 0 :: # :: 1 :: # :: 1 :: # :: 1 :: nil)
:: (^ :: 2 :: ^ :: 0 :: ^ :: 0 :: ^ :: 1 :: ^ :: nil)
:: (3 :: # :: 0 :: # :: 1 :: # :: 1 :: # :: 1 :: nil)
:: (^ :: 2 :: ^ :: 0 :: ^ :: 1 :: ^ :: 0 :: ^ :: nil)
:: (3 :: # :: 3 :: # :: 3 :: # :: 1 :: # :: 1 :: nil)
:: (^ :: 0 :: ^ :: 2 :: ^ :: 1 :: ^ :: 1 :: ^ :: nil)
:: (2 :: # :: 2 :: # :: 2 :: # :: 0 :: # :: 1 :: nil)
:: (^ :: 2 :: ^ :: 1 :: ^ :: 1 :: ^ :: 1 :: ^ :: nil) :: nil
: list (list nat)
```

Figure 10 : Execution result ($n = m = 4$). # denotes a tile and ^ corresponds to a corner.

Brick corner Wang tile

Brick corner Wang tile is a class of sets of Wang tiles. Any tile in a brick corner Wang tile set satisfy one of the following conditions:

- The top and bottom is a same color and the left and right are different.
- The left and right is a same color and the top and bottom are different.

The followings are examples of correct examples and incorrect ones.

Correct: Incorrect:

We can apply a brick corner Wang tile for wall pattern textures. For example, we consider a finite set of real numbers between 0 and 1 as a color set. (eg. {0.2, 0.4, 0.6, 0.8})

Then, a colored tile is considered as follows:

Example: left = right = 0.6, top = 0.8 and bottom = 0.4

- Since left = right, we consider a line between (0,0.6) and (1,0.6).
- Consider a perpendicular line from (0.8,1) and (0.4,0) to the horizontal line and the tile is like (Figure 4).

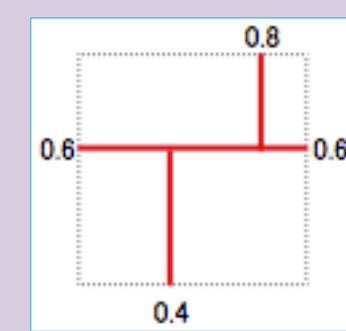


Figure 4 : Tile with lines.

(Figure 5) is an example of a Wang tiling with a set of Wang tile with the color set {0.2, 0.4, 0.6, 0.8}.

Red horizontal lines and vertical lines can be considered as a stack of bricks with different sizes. Aperiodicity of tiling can be applied to make a natural images in computer graphics.

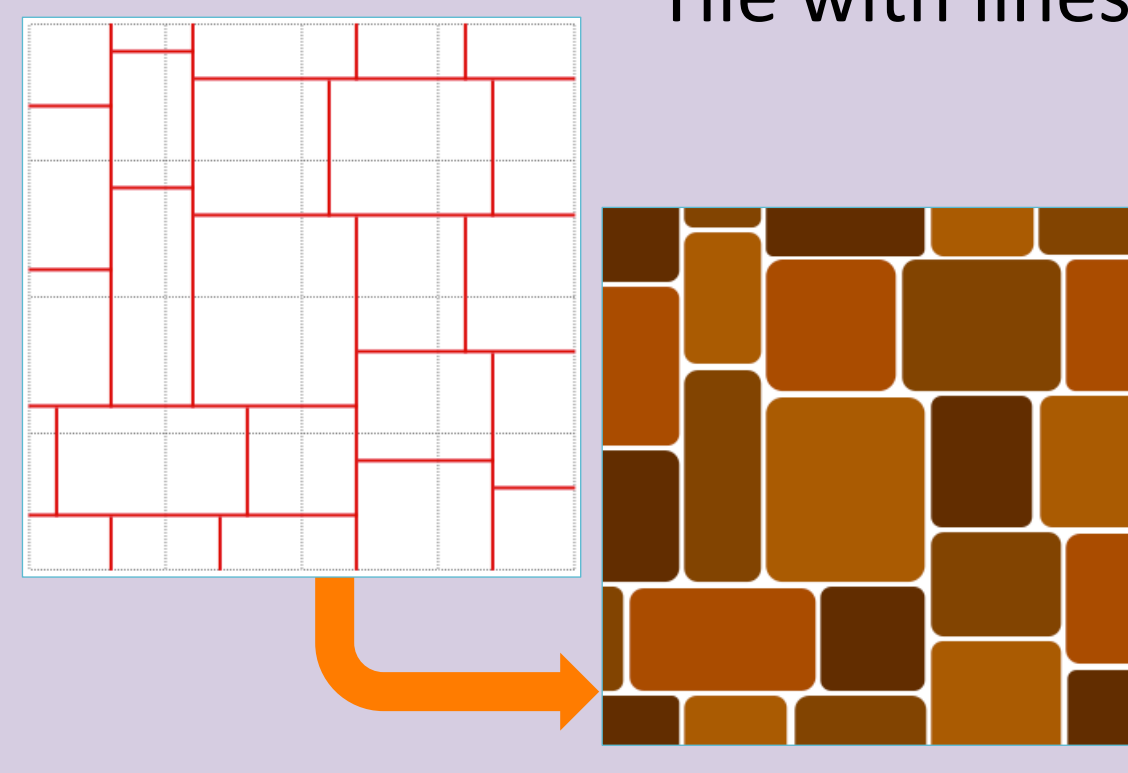


Figure 5 : We can make a design of a stack of bricks using Wang tiling.

Our tiling program

Consider a rectangle region of size $n \times m$, where n and m are natural numbers, and a boundary coloring as an initial condition (Figure 6).

We assume below 2 facts:

1. A set of color has more than 3 colors.
 - There is no brick corner Wang tile set for a 1 color set.
 - There exists an untileable boundary condition for a 2 color set.
2. We consider a rectangle $m \times n$ where $m \geq 2$ and $n \geq 2$.
 - There exists an untileable boundary condition for rectangles $m \times 1$ and $1 \times n$.

We use a set of natural numbers such as $\{i \mid i \leq k\}$ for a color set. Therefore, 0, 1 and 2 are always in a color set.

We use the type *nat* for n and m , $nat \rightarrow nat \rightarrow nat$ for boundary conditions. We note that *nat* means natural numbers $\{0, 1, 2, \dots\}$, and

$nat \rightarrow nat \rightarrow nat \cong (nat \times nat) \rightarrow nat$.

This program returns two functions, horizontal edges $e : nat \rightarrow nat \rightarrow nat$ and vertical edges $e' : nat \rightarrow nat \rightarrow nat$.

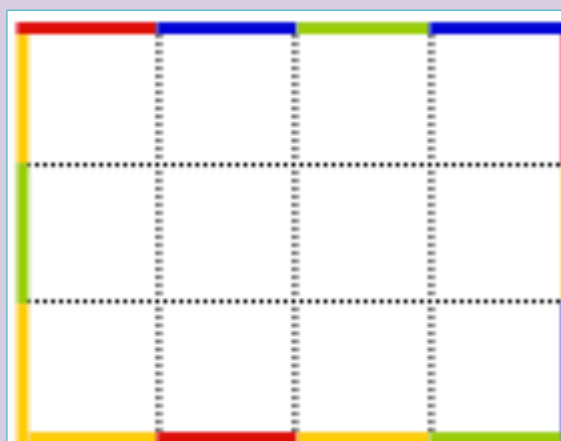


Figure 6 : Example of a boundary coloring (3 x 4 rectangle).

Correctness of our program

An implemented function in Coq can be verified a correctness of its properties.

A formulation of its properties are as follows:

- correctness of 2×2 tiling (408 lines)
- induction steps for horizontal lines (109 lines) (If $2 \times m$ is tileable, then $2 \times (m+1)$ is also tileable)
- induction steps for vertical lines (117 lines) (If $n \times m$ is tileable, then $(n+1) \times m$ is also tileable)

For a brick corner tile set with a color set which contains more than 3 colors and a rectangle including 2×2 regions, we can prove the correctness of our program. This result assures the existence of a tiling of brick corner Wang tiles for any size of rectangle including 2×2 region (Figure 11).

```
Theorem Tileable_exists :
(forall (b : boundary)(n m : nat),
2 <= n -> 2 <= m ->
Tileable n m b (e_nm n m b) (e'_nm n m b)) ->
forall (b : boundary)(n m : nat),
2 <= n -> 2 <= m ->
exists (e e' : edge), Tileable n m b e e'.
Proof.
move => H b n m H0 H1.
exists (e_nm n m b).
exists (e'_nm n m b).
apply (H b n m H0 H1).
Qed.
```

Figure 11 : The Coq proof of the existence of tiling.

References

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Future works

- Other kinds of certifications for generalized or specified brick corner tile set.
- Other kinds of certifications for a generalized region which is not a rectangle.
- Proofs for properties of a result tiling patterns such as randomness of patterns.